Math 206B Lecture 24 Notes

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1 Topics on Skew Tableau and Skew Schur Functions

1.1 Complexity of the number of skew tableau

Recall that **skew Schur functions** satisfy

$$s_{\lambda\setminus\mu} = \sum_{A \in \mathrm{SSYT}(\lambda\setminus\mu)} x^A.$$

Let $f^{\lambda\setminus\mu}$ be the number of $\mathrm{SSYT}(\lambda \setminus \mu)$.

Theorem 1.1 (Jacoi-Trudy)**.**

$$f^{\lambda\setminus\mu} = n! \det\left[\left(\frac{1}{(\lambda_i - \mu_i + j - i)!}\right)\right]$$

Corollary 1.1. *$f^{\lambda\setminus\mu}$ can be computed in polynomial time.*

Theorem 1.2 (Pittmer-Pak)**.** *Let D be an arbitrary shape of a diagram (not necessarily a skew shape). Then f^D is $\#P$ complete.*

This is equivalent to the following theorem.

Definition 1.1. $\mathrm{Bruhat}(\sigma) = \{\omega \in S_n : \omega \leq \sigma\}$.

Theorem 1.3. *$|\mathrm{Bruhat}(\sigma)|$ is $\#P$ complete.*

1.2 Littlewood-riichardson coefficients for skew Schur functions

Recall that $s_\mu s_\nu = \sum_{|\lambda|=n} c^\lambda_{\mu,\nu} s_\lambda$, where $c^\lambda_{\mu,\nu}$ are the Littlewood-Richardson coefficients.

Theorem 1.4.

$$s_{\lambda\setminus\mu} = \sum_{|\nu|=n-k} c^\lambda_{\mu,\nu} s_\nu.$$

Proof. The idea is to perform jeu-de-taquin on the skew tableau A of shape $\lambda \setminus \nu$. $\square$

One interpretation of this is that we can use Schur functions to construct $s_{\lambda\setminus\nu}$, which are symmetric functions that end up being nice.

Recall that

$$\begin{aligned} c^{\lambda}_{\mu,\nu} &= \left\langle S^{\lambda}, S^{\mu} \otimes S^{\nu} \uparrow^{S_n}_{S_k \times S_{n-k}} \right\rangle \\ &= \left\langle S^{\lambda} \downarrow^{S_n}_{S_k \times S_{n-k}}, S^{\mu} \otimes S^{\nu} \right\rangle . \end{aligned}$$

by Frobenius reciprocity. Then if π_λ is an irreducible representation of $\mathrm{GL}(n, \mathbb{C})$ corresponding to λ and s_λ is a character of π_λ, then

$$c^{\lambda}_{\mu,\nu} = \langle \pi_\lambda, \pi_\mu \otimes \pi_\nu \rangle .$$

Theorem 1.5 (Knutsen-Tao, c. 2000)**.** *For all k, $c^{\lambda}_{\mu,\nu} > 0$ iff $c^{k\lambda}_{k\mu,k\nu} > 0$.*

Corollary 1.2. *It can be decided if $c^{\lambda}_{\mu,\nu} = 0$ in polynomial time.*

Proof. $c^{\lambda}_{\mu,\nu} \neq 0$ if there exists a rational point in some certain polytope $P(\lambda, \mu, \nu)$ containing $c^{\lambda}_{\mu,\nu}$ integer points. $\square$

Theorem 1.6. *Computing $c^{\lambda}_{\mu,\nu}$ is $\#P$ complete.*

This is related to the following problem. When does $A + B = C$, when A, B, C are Hermitian matrices?

Theorem 1.7 (Klyacko)**.** *Let A, B, C be hermition matrices with (vector of) eigenvalues μ, ν, λ, respectively. Then there exist matrices A, B, C solving $A + B = C$ iff $c^{\lambda}_{\mu,\nu} \neq 0$.*